Linear transport equations valid for arbitrary collisionality: Comparison with the Chapman-Enskog expansion

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Recently we proposed a method to solve the perturbed Boltzmann equation modeled by the Bhatnagar-Gross-Krook operator [Phys. Rev. E **74**, 041204 (2006)]. In this work we use this method to derive linear transport equations in the whole collisionality range. A comparison of the closure relations derived up to the third order in the Knudsen number (super-Burnett) yields the same results as the Chapman-Enskog expansion. The contribution of the projection operators to the transport is investigated. It is pointed out that their contributions are not negligible in the super-Burnett equations and very significant in the collisionless range. The test of stability of the super-Burnett equations is also performed. It is shown that the stability problem can be related to the positivity of the generalized transport coefficients. Using the Padé approximants, nonlocal transport coefficients are proposed which present the desirable stability properties.

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I. INTRODUCTION

Equations of fluid dynamics (EFD) provide a convenient reduced description for many problems and are frequently more amenable to analytic insight, nonlinear analysis, or computational solution than the full kinetic equations. The most significant EFD are the Navier-Stokes equations (NSE) which are very well established in fluid dynamics. However it is well known that the NSE are unable to model the kinetic effects in hydrodynamic systems. The kinetic effects are not negligible when the Knudsen number corresponding to the ratio of the mean free path to the characteristic macroscopic length scale is not too small. For instance the NSE give poor results in describing the shock wave phenomena (the thickness of a shock wave is generally of the order of the mean free path) and the flow phenomena in rarefied gases.

To remedy deficiencies of the NSE in the kinetic range, much effort has been paid on the derivation of higher order hydrodynamic equations. The derivation of the so-called generalized hydrodynamic models is based mainly on the Chapman-Enskog [1] (CE) expansion around the equilibrium state, up to the third in the Knudsen number, and the Grad's moment method [2]. The most popular generalized hydrodynamic models are the Burnett equations (BE) and the super-Burnett equations (SBE) and the 13-moment equations. The NSE are of second order in space and the SBE are of fourth order. The SBE are good as long as the third-order terms are small but it is well known that they give rise to exponential instability for sufficiently small macroscopic length scale.

Several works (see, for instance, Refs. [3–12] and references therein) are reported in the literature to improve the Burnett and the super-Burnett models. They are based on the computation of additional terms to stabilize the equations with the use of the CE and the moment methods.

Because of its simplicity compared to the full integrodifferential Boltzmann equation, the Bhatnagar-Gross-Krook (BGK) kinetic equation is widely used in the kinetic theory of gases. The main shortcoming of the BGK model is that it yields just qualitative results. In particular in the standard BGK model defined by a constant collision frequency, the particles collide with the same rate in contrast to the Boltzmann collision operator which shows that the fast particles collide more often than the slow particles. It results that the computation of the Prandtl number corresponding to the ratio of the thermal conductivity to the viscosity coefficient gives $\mathrm{Pr}=1$ with the use of the BGK model instead of $\mathrm{Pr}\approx 2/3$ predicted by the full Boltzmann equation. We note that to recover the correct Prandtl number the standard BGK model was improved in the literature with the use of an anisotropic Gaussian as a reference distribution function [the ellipsoidal-statistical (ES) -BGK model] and a velocity-dependent collision frequency. Both BGK and ES-BGK models present the following desirable properties: (i) the system relaxes to the equilibrium represented by the Maxwellian; (ii) the models fulfill the reliable laws of conservation of mass, momentum, and energy.

The aim of this paper is to calculate the semicollisional transport coefficients in neutral gases and to estimate the role of the projection operators on the transport. In particular it is found that these operators affect the transport beyond the second order in the Knudsen number. On the other hand as a benchmark test for our theoretical method (called hereafter POM for projection operator method) we compare our results with the ones derived from the CE expansion. We will see that we have obtained a perfect agreement between the two results and this corroborates the exactness of our method.

The stability of the EFD derived with both approaches is also studied. Transport coefficients constructed with the Padé approximants and that ensure the stability properties are also proposed.

The paper is organized as follows. In Sec. II we present the POM and the CE methods together with the comparison of these two methods up to the third order in the Knudsen number. Section III is devoted to the contribution of the projection operators to the transport. Both the third order in the Knudsen number in the collisional range and the collisionless limit are investigated. Section IV deals with the linear

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stability of the nonlocal EFD and we present in a last section the summary of this work.

II. PROJECTION OPERATOR AND CHAPMAN-ENSKOG METHODS

A. Projection operator method

Following the kinetic model reported in Ref. [13], we consider the case of a monatomic gas without external force, modeled with the standard BGK-Boltzmann equation

$$\frac{\partial f_g}{\partial t}+\vec{v}\cdot\frac{\partial f_g}{\partial \vec{r}}=-\nu(f_g-f_{gM}), \tag{1}$$

where ν is the collision frequency, f_{gM} is the local Maxwellian, and f_g the gas distribution function (DF). The other variables are the time t, the position $\vec{r}$, and the velocity $\vec{v}$. In the most general case the collision frequency may depend on the temperature and on the particle velocity. In this work we consider a constant collision frequency; this model allows ensuring the conservative properties of the collision operator. Moreover we limit for simplicity our analysis to the one-dimensional approximation in space, i.e., $f_g=f_g(\vec{v},x,t)$ and to systems with small deviations from the equilibrium (linear analysis) represented by the global Maxwellian

$$F_M(c)=\mu_0\exp\left(-\frac{c^2}{2v_t^2}\right), \tag{2}$$

where $\mu_0=\frac{n_0}{(2\pi)^{3/2}v_t^3}$, $\vec{c}$ is the peculiar velocity, $v_t=(T_0/m)^{1/2}$ is the thermal velocity, m is the mass of one particle, and n_0 and T_0 (in energy units) are the background density and temperature, respectively. From Eq. (1) the perturbed state is described by the following kinetic equation,

$$\frac{\partial f}{\partial t}+v_x\frac{\partial f}{\partial x}=-\nu(f-f_M), \tag{3}$$

where f is the perturbed distribution function and

$$f_M(c,x,t)=\frac{n(x,t)}{n_0}F_M(c)+\frac{T(x,t)}{T_0}\left(\frac{c^2}{2v_t^2}-\frac{3}{2}\right)F_M(c) \tag{4}$$

is the perturbed Maxwellian defined by the perturbed density $n(x,t)$ and the perturbed temperature $T(x,t)$. In addition the collision operator in Eq. (3) must verify the conservative properties

$$\int \nu(f-f_M)\left[1,m\vec{v},\frac{mv^2}{2}\right]d^3v=0. \tag{5}$$

Following the Braginskii [14] approach we express Eq. (3) as a function of the peculiar velocity $\vec{c}$ making the variable change ($\vec{v}\rightarrow\vec{c}$) and keeping the spatial variable x defined in the laboratory frame,

$$\frac{\partial f(\vec{c},x,t)}{\partial t}+c_x\frac{\partial f(\vec{c},x,t)}{\partial x}-\frac{\partial V}{\partial t}\frac{\partial F_M(c)}{\partial v_x}=-\nu[f(\vec{c},x,t)-f_M(c,x,t)], \tag{6}$$

where $V(x,t)$ is the local hydrodynamic velocity. To solve Eq. (6) we proceed as follows. First we perform the spatial and temporal Fourier transforms using, respectively, the transform variables k and ω,

$$-i\omega f+ikc_xf+\nu f=\nu f_M+i\omega\frac{c_x}{v_t^2}F_MV-ik\frac{c_x^2}{v_t^2}F_MV, \tag{7}$$

second, we expand $f(\vec{c},k,\omega)$ and Eq. (7) on the Legendre polynomial basis obtaining, respectively,

$$f(\vec{c},\omega,k)=\sum_{n=0}^{\infty}P_n(\mu)f_n(y,\omega,k) \tag{8}$$

and

$$-i\omega f_0+ikv_t\sqrt{2/3}y^{1/2}f_1+\nu f_0=\nu f_M-\frac{2}{3}ikv_tyF_M\frac{V}{v_t}, \tag{9}$$

$$-i\omega f_1+ikv_ty^{1/2}\sqrt{\frac{2}{3}}f_0+ikv_t\frac{2\sqrt{2}}{\sqrt{15}}y^{1/2}f_2+\nu f_1=i\omega\frac{V}{v_t}\sqrt{\frac{2}{3}}y^{1/2}F_M, \tag{10}$$

$$-i\omega f_2+\frac{2\sqrt{2}}{\sqrt{15}}ikv_ty^{1/2}f_1+\nu f_2+\frac{3\sqrt{2}}{\sqrt{35}}ikv_ty^{1/2}f_3=-\frac{4}{3\sqrt{5}}y\mu_0\exp(-y)ikV, \tag{11}$$

$$-i\omega f_{n+1}+\frac{n+1}{\sqrt{(2n+1)(2n+3)}}ik\sqrt{2}v_ty^{1/2}f_n+\nu f_{n+1}+\frac{n+2}{\sqrt{(2n+3)(2n+5)}}ik\sqrt{2}v_ty^{1/2}f_{n+2}=0\quad(n>1), \tag{12}$$

where $P_n(\mu)$ is the Legendre polynomial of order n, $\mu=\frac{c_x}{c}$, and $y=mc^2/2T_0$. We note that the role of the collision operator can be split into two parts. The anisotropic part of this operator in Eqs. (10)–(12) tends to reduce the anisotropies $f_{n\geq 1}$ and the isotropic part in Eq. (9) relaxes f_0 toward the perturbed Maxwellian f_M. It is obvious that in the nonequilibrium state f_0 is not a Maxwellian and it has to be treated on equal footing as the anisotropic components of the DF, $f_{n\geq 1}$.

We can now go a step further and use the techniques of Ref. [15] to solve the infinite set of equations (12) with the use of the continued fractions, obtaining for the first and the second anisotropic distribution functions the following expressions:

$$f_1=-\sqrt{\frac{2}{3}}v_ty^{1/2}F_1ikf_0-\frac{8\sqrt{2}}{15\sqrt{3}}v_ty^{3/2}F_1F_2F_Mk^2V+\frac{\sqrt{2}}{\sqrt{3}v_t}y^{1/2}F_1F_Mi\omega V, \tag{13}$$

$$f_2 = -\frac{4}{3\sqrt{5}}v_t^2 y F_1 F_2 k^2 f_0 - \frac{4}{3\sqrt{5}}\nu y F_1 F_2 F_M ikV. \quad (14)$$

Here F_n are continued fractions defined by the recursion relation

$$F_n = \left[-i\omega + \nu + \frac{(n+1)^2}{4(n+1)^2-1}2k^2 v_t^2 y F_{n+1}\right]^{-1}, \quad (15)$$

which incorporates the contributions from all the Legendre modes.

Now we use the projection operator techniques which can be used in kinetic theory [16–20]. We define the projection operator P,

$$P[\nu(f-f_M)] = 0, \quad (16)$$

and Q, its orthogonal complement. These operators are calculated for the BGK operator in Ref. [13] and we just recall their expressions

$$P[h(y)] = \frac{2}{\sqrt{\pi}}\left[\frac{5}{2}\int_0^\infty y^{1/2}h(y)dy - \int_0^\infty y^{3/2}h(y)dy\right]\exp(-y) + \frac{4}{3\sqrt{\pi}}\left[-\frac{3}{2}\int_0^\infty y^{1/2}h(y)dy + \int_0^\infty y^{3/2}h(y)dy\right]y\exp(-y) \quad (17)$$

and

$$Q = 1 - P, \quad (18)$$

where $h(y)$ is an arbitrary isotropic function. These operators are used to separate the Boltzmann Eq. (3) into

$$P\left[\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x}\right] = 0 \quad (19)$$

and

$$Q\left[\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x}\right] = -\nu(f-f_M). \quad (20)$$

We point out that Eq. (19) expresses at the kinetic level, the balance equations with vanishing collision terms, namely, the continuity equation, the momentum balance equation (or the motion equation), and the energy balance equation. Using Eqs. (9) and (19), we find the isotropic equation which includes the conservative laws,

$$-i\omega(f_0-f_M) + ikv_t\sqrt{\frac{2}{3}}y^{1/2}f_1 - \sqrt{\frac{8}{3\pi}}ikv_t\left(\frac{2y}{3}-1\right)\exp(-y)\int_0^\infty y^2 f_1 dy = \nu(f_M - f_0). \quad (21)$$

We should note here that the use of the projection operators is equivalent to the use of the initial conditions as performed by Brantov *et al.* in Ref. [21]. Indeed if we use, instead of the Fourier transform in time, the Laplace transform to take into account the initial conditions, Eq. (9) becomes

$$-i\omega f_0 + ikv_t\sqrt{2/3}y^{1/2}f_1 + \nu f_0 = \nu f_M - \frac{2}{3}ikv_t y F_M \frac{V}{v_t} + f_M(k,c,t=0), \quad (22)$$

where it is supposed that the initial DF is the perturbed Maxwellian

$$f_M(k,c,t=0) = \left[\frac{n(t=0)}{n_0} + \left(y - \frac{3}{2}\right)\frac{T(t=0)}{T_0}\right]F_M. \quad (23)$$

Multiplying Eq. (23) by $y^{1/2}$ and $y^{3/2}$, integrating upon the y variable, and using the conservative properties

$$\int_0^\infty y^{1/2} f_0 dy = \mu_0 \frac{\sqrt{\pi}}{2}\frac{n}{n_0},$$

$$\int_0^\infty y^{3/2} f_0 dy = \mu_0 \frac{3\sqrt{\pi}}{4}\left(\frac{n}{n_0} + \frac{T}{T_0}\right),$$

we readily obtain the expression of the initial hydrodynamics

$$\frac{n(t=0)}{n_0} = -i\omega\frac{n}{n_0} + ikv_t\frac{V}{v_t}, \quad (24)$$

$$\frac{T(t=0)}{T_0} = -i\omega\frac{T}{T_0} + ikv_t\frac{2}{3}\frac{V}{v_t} + ikv_t\frac{4\sqrt{2}}{3\sqrt{3\pi}\mu_0}\int_0^\infty y^2 f_1 dy. \quad (25)$$

Substituting expressions (24) and (25) into Eq. (22), we recover exactly Eq. (21). This equivalence between the two approaches shows in particular that the collisionless range which is usually described as an initial value problem is well described by our method. We note in addition that the collisionless range can be understood [22] as a collision regime in the limit, $\nu \to 0$.

To derive the isotropic DF as a function of the driven forces, we substitute expression (13) into Eq. (21),

$$-i\omega(f_0-f_M) + \frac{2}{3}k^2 v_t^2 y F_1 f_0 + \frac{2}{3}ikv_t i\omega y F_1 F_M \frac{V}{v_t} - \frac{16}{45}k^2 v_t^2 y^2 F_1 F_2 F_M ikv_t \frac{V}{v_t} - \sqrt{\frac{8}{3\pi}}ikv_t\left(\frac{2y}{3}-1\right)\exp(-y)\int_0^\infty y^2 f_1 dy = \nu(f_M - f_0). \quad (26)$$

Thus, Eqs. (13), (14), and (26) are the exact solutions of the Boltzmann equation up to the second anisotropy. From these equations we can calculate the relevant transport quantities to close the three hydrodynamic equations.

B. Chapman-Enskog method

We present in this section the well-known CE method applied to the BGK equation. This method should be applied to solve any kinetic equation characterized by the assumption that the driven forces, i.e., external fields and gradients (including variation in time), are small compared to the effects of the collision by means of a parameter $\varepsilon \ll 1$. Formally it results a modified BGK equation,

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} = \frac{1}{\varepsilon}[-\nu(f - f_M)]. \tag{27}$$

The CE method consists to expand the distribution function, the heat flux q_x, and the stress Π_{xx}, in a series of the formal parameter, $\varepsilon \ll 1$,

$$f = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \cdots, \tag{28}$$

$$q_x = \varepsilon q_x^1 + \varepsilon^2 q_x^2 + \cdots, \tag{29}$$

$$\Pi_{xx} = \varepsilon \Pi_{xx}^1 + \varepsilon^2 \Pi_{xx}^2 + \cdots, \tag{30}$$

where f^0, f^1, f^2, q_x^1, q_x^2, Π_{xx}^1, Π_{xx}^2, etc. are driven by the local gradients and the parameter ε plays the role of the Knudsen number, i.e., $K = \frac{kv_t}{\nu} \sim \varepsilon$. In addition the CE method is based on the hypothesis that the following relations are fulfilled:

$$\int f^0 d^3v = n, \quad \int \vec{v} f^0 d^3v = n\vec{V}, \quad \int \frac{m(\vec{v} - \vec{V})^2}{2} f^0 d^3v = \frac{3}{2} nT \tag{31}$$

and for $\alpha > 0$,

$$\int \left(1, m\vec{v}, \frac{mv^2}{2}\right) f^\alpha d^3v = 0. \tag{32}$$

Let us compute the solution of Eq. (27) up to the third order to derive the transport coefficients of the NSE, the BE, and the SBE. To do so, we use expansion (28) up to the third order. In addition we expand the DF on the Legendre polynomials basis in order to extract the first and the second anisotropic DF,

$$f = \sum_{n=0}^{\infty} f_n^0(y) P_n(\mu) + \sum_{n=0}^{\infty} f_n^1(y) P_n(\mu) + \sum_{n=0}^{\infty} f_n^2(y) P_n(\mu) + \sum_{n=0}^{\infty} f_n^3(y) P_n(\mu). \tag{33}$$

The starting equation is Eq. (27) written in the Fourier space,

$$\varepsilon\left[-i\omega + ikv_t\sqrt{\frac{2}{3}} y^{1/2} P_1(\mu)\right] f = -\nu(f - f_M). \tag{34}$$

(i) Zeroth order—Euler equations: Keeping the lower order with respect to the Knudsen number we obtain

$$f^0 = f_M = \left[\frac{n}{n_0} + \left(y - \frac{3}{2}\right)\frac{T}{T_0} F_M\right] P_0(\mu) + \left(\sqrt{\frac{2}{3}} y^{1/2} \frac{V}{v_t} F_M\right) P_1(\mu), \tag{35}$$

which corresponds to vanishing heat flux and the stress tensor ($q_x^0 = 0$ and $\Pi_{xx}^0 = 0$).

(ii) First order—Navier-Stokes equations: The first-order part of Eq. (34) is

$$\left[-i\omega + ikv_t\sqrt{\frac{2}{3}} y^{1/2} P_1(\mu)\right]^1 f^0 = -\nu f^1, \tag{36}$$

where the superscript in the left-hand side means the order of the temporal operator $(-i\omega)$ in the EFD. Using the recursion relation

$$P_1(\mu) P_n(\mu) = \frac{\sqrt{3}}{\sqrt{2n+1}}\left[\frac{n+1}{\sqrt{2n+3}} P_{n+1} + \frac{n}{\sqrt{2n-1}} P_{n-1}\right] \tag{37}$$

and the lower order EFD

$$(i\omega)^0 \frac{n}{n_0} = ikV, \tag{38}$$

$$(i\omega)^0 \frac{V}{v_t} = \left(ikv_t \frac{n}{n_0} + ikv_t \frac{T}{T_0}\right), \tag{39}$$

and

$$(i\omega)^0 \frac{T}{T_0} = \frac{2}{3} ikV, \tag{40}$$

we readily deduce

$$f^1 = \left\{-\frac{ikv_t}{\nu}\sqrt{\frac{2}{3}} y^{1/2}\left(y - \frac{5}{2}\right) F_M \frac{T}{T_0}\right\} P_1(\mu) + \left\{-\frac{ikv_t}{\nu} \frac{4}{3\sqrt{5}} y F_M \frac{V}{v_t}\right\} P_2(\mu). \tag{41}$$

The heat flux and the stress are defined at each order "i," by the first and the second anisotropy as follows:

$$q_x^i = (2\pi)^{3/2}\sqrt{\frac{8}{3\pi}} T_0 v_t^4 \int_0^\infty y^2 f_1^i dy, \tag{42}$$

$$\Pi_{xx}^i = (2\pi)^{3/2} \frac{8}{3\sqrt{5\pi}} T_0 v_t^3 \int_0^\infty y^{3/2} f_2^i dy. \tag{43}$$

Inserting expression (41) into Eqs. (42) and (43) we obtain

$$q_x^1 = -\frac{5}{2} ik \frac{n_0 T_0 v_t^2}{\nu} \frac{T}{T_0}, \tag{44}$$

$$\Pi_{xx}^1 = -\frac{4}{3} ik \frac{n_0 T_0 v_t}{\nu} \frac{V}{v_t}. \tag{45}$$

(iii) Second order—Burnett equations: The corresponding kinetic equation is

$$\left[-i\omega + ikv_t\sqrt{\frac{2}{3}} y^{1/2} P_1(\mu)\right]^1 f^1 + \left[-i\omega + ikv_t\sqrt{\frac{2}{3}} y^{1/2} P_1(\mu)\right]^2 f^0 = -\nu f^2. \tag{46}$$

Using the first-order EFD, i.e.,

$$(-i\omega)^1 \frac{n}{n_0} = 0, \tag{47}$$

$$(-i\omega)^1\frac{V}{v_t} = -\frac{1}{mn_0}ik\Pi_{xx}^1, \tag{48}$$

and

$$(-i\omega)^1\frac{T}{T_0} = -\frac{2}{3n_0}ikq_x^1, \tag{49}$$

we obtain

$$f^2 = \left\{-\frac{k^2v_t^2}{\nu^2}\left(\frac{2}{3}y^2 - \frac{10}{3}y + \frac{5}{2}\right)F_M\frac{T}{T_0}\right\}P_0(\mu) + \left\{\sqrt{\frac{2}{3}}\frac{k^2v_t^2}{\nu^2}\left(\frac{2}{15}y^{3/2} - \frac{1}{3}y^{1/2}\right)F_M\frac{V}{v_t}\right\}P_1(\mu) + \left\{\frac{k^2v_t^2}{\nu^2}F_M\left[\frac{2}{3}y\frac{n}{n_0} + \left(-\frac{2}{3}y^2 + \frac{7}{3}y\right)\frac{T}{T_0}\right]\right\}P_2(\mu). \tag{50}$$

The corresponding transport quantities are

$$q_x^2 = \frac{1}{3}\frac{k^2v_t^2}{\nu^2}n_0T_0v_t\frac{V}{v_t}, \tag{51}$$

$$\Pi_{xx}^2 = \frac{4}{3}\frac{k^2v_t^2}{\nu^2}n_0T_0\frac{n}{n_0}. \tag{52}$$

(iv) Third order—super-Burnett equations: The Boltzmann Eq. (34) expanded on the third order reads

$$\left[-i\omega + ikv_t\sqrt{\frac{2}{3}}y^{1/2}P_1(\mu)\right]^1 f^2 + \left[-i\omega + ikv_t\sqrt{\frac{2}{3}}y^{1/2}P_1(\mu)\right]^2 f^1 + \left[-i\omega + ikv_t\sqrt{\frac{2}{3}}y^{1/2}P_1(\mu)\right]^3 f^0 = -\nu f^3. \tag{53}$$

Following the same procedure as above we write

$$(-i\omega)^2\frac{n}{n_0} = 0, \tag{54}$$

$$(-i\omega)^2\frac{V}{v_t} = -\frac{1}{mn_0}ik\Pi_{xx}^2, \tag{55}$$

and

$$(-i\omega)^2\frac{T}{T_0} = -\frac{2}{3n_0}ikq_x^2, \tag{56}$$

and after some algebra we find

$$f^3 = \left\{\frac{ik^3v_t^3}{\nu^3}\left(-\frac{8}{15}y^2 + \frac{24}{9}y - 2\right)F_M\frac{V}{v_t}\right\}P_0(\mu) + \left\{\frac{ik^3v_t^3}{\nu^3}F_M\sqrt{\frac{2}{3}}\left[\left(\frac{6}{5}y^{5/2} - \frac{101}{15}y^{3/2} + \frac{19}{3}y^{1/2}\right)\frac{T}{T_0} + \left(-\frac{2}{5}y^{3/2} + y^{1/2}\right)\frac{n}{n_0}\right]\right\}P_1(\mu) + \left\{\frac{ik^3v_t^3}{\nu^3}\frac{2}{\sqrt{5}}\left(-\frac{4}{21}y^2 + \frac{14}{9}y\right)F_M\frac{V}{v_t}\right\}P_2(\mu). \tag{57}$$

We deduce finally the transport expressions of the third order,

$$q_x^3 = \frac{25}{6}\frac{ik^3v_t^3}{\nu^3}n_0T_0v_t\frac{T}{T_0} - \frac{ik^3v_t^3}{\nu^3}n_0T_0v_t\frac{n}{n_0}, \tag{58}$$

$$\Pi_{xx}^3 = \frac{16}{9}\frac{ik^3v_t^3}{\nu^3}n_0T_0\frac{V}{v_t}. \tag{59}$$

C. Comparison between the POM and the CE method

It is well known that the CE expansion is formally exact and we use the results derived above as a benchmark test to corroborate the exactness of our results. To compare our results with the ones calculated in Sec. II B, we expand solutions (13), (14), and (26) up to the third order in the Knudsen number. For this we expand the continued fractions up to the second order in the Knudsen number,

$$F_n \approx \frac{1}{\nu}\left[1 + i\frac{\omega}{kv_t}K - \frac{\omega^2}{k^2v_t^2}K^2\left(1 + \frac{2(n+1)^2}{4(n+1)^2-1}\frac{k^2v_t^2}{\omega^2}y\right)\right]. \tag{60}$$

Using expansion (60) into Eqs. (13), (14), and (26), we derive the first and the second anisotropic DF as functions of the driven forces and thus we deduce the heat flux q_x^{POM} and the stress Π_{xx}^{POM},

$$\frac{q_x^{POM}(k,\omega)}{n_0T_0v_t} = -\frac{5}{2}\frac{ikv_t}{\nu}\frac{n}{n_0} + \frac{5}{2}\frac{i\omega kv_t}{\nu^2}\frac{n}{n_0} + \frac{5}{2}\frac{i\omega^2kv_t}{\nu^3}\frac{n}{n_0} + \frac{14}{3}\frac{ik^3v_t^3}{\nu^3}\frac{n}{n_0} - 5\frac{ikv_t}{\nu}\frac{T}{T_0} + 5\frac{i\omega kv_t}{\nu^2}\frac{T}{T_0} + 5\frac{i\omega^2kv_t}{\nu^3}\frac{T}{T_0} + \frac{52}{3}\frac{k^3v_t^3}{\nu^3}\frac{T}{T_0} + \frac{5}{2}\frac{i\omega}{\nu}\frac{V}{v_t} - \frac{5}{2}\frac{\omega^2}{\nu^2}\frac{V}{v_t} - \frac{14}{3}\frac{k^2v_t^2}{\nu^2}\frac{V}{v_t} - \frac{5}{2}\frac{i\omega^3}{\nu^3}\frac{V}{v_t} - 14\frac{i\omega k^2v_t^2}{\nu^3}\frac{V}{v_t}, \tag{61}$$

$$\frac{\Pi_{xx}^{POM}(k,\omega)}{n_0T_0} = -\frac{4}{3}\frac{k^2v_t^2}{\nu^2}\frac{n}{n_0} + \frac{8}{3}\frac{ik^3v_t^3}{\nu^3}\frac{V}{v_t} - \frac{8}{3}\frac{k^2v_t^2}{\nu^2}\frac{T}{T_0} - \frac{ikv_t\omega^2}{\nu^3}\frac{n}{n_0} - \frac{28}{15}\frac{ik^3v_t^3}{\nu^3}\frac{n}{n_0} - 5\frac{ikv_t}{\nu}\frac{T}{T_0} + 5\frac{kv_t\omega}{\nu^2}\frac{T}{T_0} + 5\frac{ikv_t\omega^2}{\nu^3}\frac{T}{T_0} + 14\frac{ik^3v_t^3}{\nu^3}\frac{T}{T_0} + \frac{5}{2}\frac{i\omega}{\nu}\frac{V}{v_t} - \frac{5}{2}\frac{\omega^2}{\nu^2}\frac{V}{v_t} - \frac{14}{3}\frac{k^2v_t^2}{\nu^2}\frac{V}{v_t} - \frac{5}{2}\frac{i\omega^3}{\nu^3}\frac{V}{v_t} - 14\frac{k^2v_t^2i\omega}{\nu^3}\frac{V}{v_t}. \tag{62}$$

Keeping the terms of expansions (61) and (62) up to the third order in K, we retrieve exactly the CE results derived above. In addition since both the CE and POM methods yield exact solutions for the distribution function, we expect that the expansion of the continued fractions in Eqs. (13), (14), and (26), at any arbitrary order, matches order by order the CE expansion of the components f_0, f_1, and f_2. We should mention, however, that the difference between our method (valid only at the linear approximation) and the CE one is that the collisional invariance is taken into account at the kinetic level with the use of the projection operators. We point out that recently Karlin *et al.* [23,24] have derived exact linear hydrodynamics from the Boltzmann equation modeled with the BGK collision operator. Their approach is based on the method of invariant manifold and the results obtained match order by order the CE expansion.

III. CONTRIBUTION OF THE PROJECTION OPERATORS

Our method is based on the technique of the projection operators which ensure the invariance of the BGK collision operator. These operators play a role only in the isotropic Eq. (26) and this has the effect of making f_0 non-Maxwellian beyond the Burnett approximation. In this section we study the role of the projection operators by imposing $f_0 = f_M$. This method is called hereafter WPOM (without the projection operator method). The basic Eqs. (13) and (14) with $f_0 = f_M$ read

$$f_1 = -\frac{\sqrt{2}}{\sqrt{3}} ik v_t y^{1/2} F_1 \left[\frac{n}{n_0}\mu_0 \exp(-y) + \frac{T}{T_0}\mu_0\left(y-\frac{3}{2}\right)\exp(-y)\right] - \frac{8\sqrt{2}}{15\sqrt{3}} v_t y^{3/2} F_1 F_2 \mu_0 \exp(-y) k^2 V + \frac{\sqrt{2}}{\sqrt{3}v_t} y^{1/2} F_1 \mu_0 \exp(-y) i\omega V, \tag{63}$$

$$f_2 = -\frac{4}{3\sqrt{5}} v_t^2 y F_1 F_2 k^2 \left[\frac{n}{n_0}\mu_0 \exp(-y) + \frac{T}{T_0}\mu_0\left(y-\frac{3}{2}\right)\exp(-y)\right]. \tag{64}$$

First we study the case of the super-Burnett order and second we investigate the collisionless limit.

A. Super-Burnett order

We expand Eqs. (63) and (64) up to the third order in the Knudsen number with the use of the expansion of the continued fractions [Eq. (60)]. Then, we express the results only with respect to the spatial variable with the use of the EFD obtaining after some algebra

$$q_x^{WPOM}(x,t) = q_x^{POM}(x,t) + \frac{20}{6}\frac{n_0 v_t^4}{\nu^3}\frac{\partial^3 T}{\partial x^3}, \tag{65}$$

$$\Pi_{xx}^{WPOM}(x,t) = \Pi_{xx}^{POM}(x,t). \tag{66}$$

We remark that this approximate method gives the same results for the stress but a non-negligible corrective term for the heat flux. If we add higher ordering in the continued fractions we should obtain more corrective terms in the expression of the heat flux and the stress. In particular it is interesting to estimate the difference between these two methods in the collisionless range where we must keep all the Legendre components of the DF. The collisionless range is essential for instance in the low pressure gases systems in laboratories, in the high altitude flight (in the outer atmosphere), and the microelectromechanical systems.

B. Collisionless limit

To compare the POM and the WPOM in the collisionless range we introduce in the gas an external force $F(x,t)$ with small amplitude directed along the x axis. The kinetic Eq. (3) and the motion equation become, respectively,

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + \frac{F(x,t)}{m}\frac{\partial F_M}{\partial v_x} = 0, \tag{67}$$

$$\frac{\partial V}{\partial t} = -\frac{T_0}{n_0 m}\frac{\partial n}{\partial x} - \frac{1}{m}\frac{\partial T}{\partial x} + \frac{F}{m} - \frac{1}{n_0 m}\frac{\partial \Pi_{xx}}{\partial x}. \tag{68}$$

This force allows the calculation of the fluid response function [25,26] in the Fourier space

$$R_f(k,\omega) = \frac{ikn(k,\omega)T_0}{n_0 F(k,\omega)}. \tag{69}$$

This function describes the response of the system due to the external force $F(k,\omega)$. To compute it we calculate from the kinetic equation, the heat flux, and the stress. We incorporate these transport quantities into the conservative EFD and we deduce the fluid response functions with the WPOM and the POM.

For the WPOM the relevant kinetic equations are

$$f_0(v) = f_M(v), \tag{70}$$

$$f_1 = -\sqrt{\frac{2}{3}} v_t y^{1/2} F_1 ik f_0 - \frac{8\sqrt{2}}{15\sqrt{3}} v_t y^{3/2} F_1 F_2 F_M k^2 V + \frac{\sqrt{2}}{\sqrt{3}} y^{1/2} F_1 F_M i\omega \frac{V}{v_t} + \frac{\sqrt{2}}{\sqrt{3}} y^{1/2} F_1 F_M \frac{F}{m} v_t, \tag{71}$$

$$f_2 = -\frac{4}{3\sqrt{5}} v_t^2 y F_1 F_2 k^2 f_0 - \frac{4}{3\sqrt{5}} \nu y F_1 F_2 F_M ikV. \tag{72}$$

Solving these equations, we readily deduce

$$q_x^{WPOM}(k,\omega) = -n_0 T_0 v_t \left(ikn\frac{v_t}{n_0}\right)\frac{4}{3\sqrt{\pi}} I_{5/2}^{0,1,0} - n_0 T_0 v_t \left(ikT\frac{v_t}{T_0}\right)\frac{4}{3\sqrt{\pi}}\left[I_{7/2}^{0,1,0} - \frac{3}{2} I_{5/2}^{0,1,0}\right] - n_0 T_0 v_t \left(\frac{V}{v_t}\right)\frac{4}{3\sqrt{\pi}}\left[\frac{8k^2 v_t^2}{15} I_{7/2}^{0,1,1} - i\omega I_{5/2}^{0,1,0}\right] + n_0 T_0 v_t \left(\frac{F}{m v_t}\right)\frac{4}{3\sqrt{\pi}} I_{5/2}^{0,1,0} \tag{73}$$

and

$$\Pi_{xx}^{WPOM}(k,\omega) = -n_0T_0\left(\frac{n}{n_0}\right)\frac{32k^2v_t^2}{45\sqrt{\pi}}I_{5/2}^{0,1,1} - n_0T_0\left(\frac{T}{T_0}\right)\frac{32k^2v_t^2}{45\sqrt{\pi}}\left[I_{7/2}^{0,1,1} - \frac{3}{2}I_{5/2}^{0,1,1}\right] - n_0T_0\frac{32}{45\sqrt{\pi}}I_{5/2}^{0,1,1}ikv_t\nu\frac{V}{v_t}, \tag{74}$$

where $I_n^{i,j,k} = \int_0^\infty y^n(F_0)^i(F_1)^j(F_2)^k\exp(-y)dy$. We note here that the continued fractions used are defined in the collisionless limit, i.e., in Eq. (15) we set $\nu=0$.

Now we calculate the transport coefficients with the use of the POM approach. This method is presented in details in Ref. [13]. The equation for the isotropic component is Eq. (26) with $\nu=0$ and the first and the second component of the DF are still given by Eqs. (13) and (14), respectively. The solution of Eqs. (13), (14), and (26) is straightforward and we obtain the following transport expressions:

$$\begin{aligned} q_x^{POM}(k,\omega) = {}& -n_0T_0v_t\left\{\frac{3\sqrt{\pi}}{2D}\left[\left(\frac{5}{2}I_{3/2}^{1,0,0} - I_{5/2}^{1,0,0}\right) - \frac{3}{2}\left(\frac{5}{2}I_{1/2}^{1,0,0} - I_{3/2}^{1,0,0}\right) + 3i\omega\right]\right\}\left[\frac{1}{2k^2v_t^2}\left(\frac{ikv_tn}{n_0} - \frac{F}{mv_t}\right)\right] - n_0T_0v_t\left\{\frac{9\sqrt{\pi}}{4D}\left[\left(\frac{5}{2}I_{1/2}^{1,0,0}\right.\right.\right. \\ & \left.\left.\left. - I_{3/2}^{1,0,0}\right) + 3i\omega\right]\right\}\left(\frac{ikv_t}{2k^2v_t^2}\frac{T}{T_0}\right) - \frac{n_0T_0v_t}{D}\left\{i\omega\left[I_{5/2}^{1,1,0}\left(\frac{5}{2}I_{1/2}^{1,0,0} - I_{3/2}^{1,0,0}\right) - I_{3/2}^{1,1,0}\left(\frac{5}{2}I_{3/2}^{1,0,0} - I_{5/2}^{1,0,0}\right)\right]\right\}\frac{V}{v_t} \\ & - \frac{n_0T_0v_t}{D}\left\{\frac{8}{15}k^2v_t^2\left[I_{5/2}^{1,1,1}\left(\frac{5}{2}I_{3/2}^{1,0,0} - I_{5/2}^{1,0,0}\right) - I_{7/2}^{1,1,1}\left(\frac{5}{2}I_{1/2}^{1,0,0} - I_{3/2}^{1,0,0}\right)\right]\right\}\frac{V}{v_t} \end{aligned} \tag{75}$$

and

$$\begin{aligned} \Pi_{xx}^{POM}(k,\omega) = {}& -\frac{16k^2v_t^2}{45D}n_0T_0\left[I_{7/2}^{1,1,1}\left(I_{3/2}^{1,0,0} - \frac{3}{2}I_{1/2}^{1,0,0}\right) - I_{5/2}^{1,1,1}\left(I_{5/2}^{1,0,0} - \frac{3}{2}I_{3/2}^{1,0,0}\right)\right]\left(\frac{n}{n_0} - \frac{F}{kmv_t^2}\right) - \frac{8k^2v_t^2}{15D}n_0T_0(I_{5/2}^{1,1,1}I_{3/2}^{1,0,0} - I_{7/2}^{1,1,1}I_{1/2}^{1,0,0})\frac{T}{T_0} \\ & - \frac{32}{45\sqrt{\pi}}n_0T_0ikv_t\left\{-\frac{2k^2v_t^2}{3}i\omega I_{7/2}^{1,2,1} - \frac{2k^2v_t^2}{3D}i\omega I_{5/2}^{1,1,1}(I_{3/2}^{1,1,0}I_{5/2}^{1,0,0} - I_{5/2}^{1,1,0}I_{3/2}^{1,0,0}) + \frac{2k^2v_t^2}{3D}i\omega I_{7/2}^{1,1,1}(I_{3/2}^{1,1,0}I_{3/2}^{1,0,0} - I_{5/2}^{1,1,0}I_{1/2}^{1,0,0})\right. \\ & \left. + \frac{8k^2v_t^2}{15D}I_{5/2}^{1,1,1}(I_{5/2}^{1,1,1}I_{5/2}^{1,0,0} - I_{7/2}^{1,1,1}I_{3/2}^{1,0,0}) - \frac{8k^2v_t^2}{15D}I_{7/2}^{1,1,1}(I_{5/2}^{1,1,1}I_{3/2}^{1,0,0} - I_{7/2}^{1,1,1}I_{1/2}^{1,0,0}) + \frac{8k^4v_t^4}{15}I_{9/2}^{1,2,2}\right\}\frac{V}{v_t}, \end{aligned} \tag{76}$$

where $D = I_{3/2}^{100}I_{3/2}^{100} - I_{5/2}^{100}I_{1/2}^{100}$. Using Eqs. (73)–(76) in the linear EFD, we deduce the expressions of the WPOM and POM fluid response functions,

$$R_f(\xi) = \left[-2\xi^2 - i\frac{k}{|k|}\sqrt{2}\xi\eta_V + (1-\eta_n) + \frac{\frac{2}{3}(1-\eta_T)[\Lambda_n + (1-\Lambda_V)i\sqrt{2}\xi]}{i\sqrt{2}\xi - \frac{2}{3}\Lambda_T}\right]^{-1}, \tag{77}$$

where the transport coefficients are defined by the expressions

$$\Pi_{xx} = \left(-\eta_n\frac{n}{n_0} - \eta_T\frac{T}{T_0} - i\frac{k}{|k|}\eta_V\frac{V}{v_t}\right)n_0T_0, \tag{78}$$

$$q_x = \left(-\Lambda_n i\frac{k}{|k|}\frac{n}{n_0} - \Lambda_T i\frac{k}{|k|}\frac{T}{T_0} - \Lambda_V\frac{V}{v_t}\right)n_0T_0v_t, \tag{79}$$

and where $\xi = \frac{\omega}{\sqrt{2}kv_t}$ is a dimensionless phase velocity. On the other hand, from Eq. (67) we can also compute the exact kinetic response function [25,26]

$$R_k(\xi) = PP\int_{-\infty}^{+\infty}\frac{z\exp(-z^2)}{z-\xi}dz + i\sqrt{\pi}\xi\exp(-\xi^2), \tag{80}$$

where PP means the Cauchy principal value. The numerical computation of the WPOM and POM fluid response functions and the kinetic response function (80) are given in Fig. 1. We remark that the kinetic response function and the POM fluid response function coincide exactly. This confirms that the approach used with the POM method is exact. In the quasistationary range $\xi \ll 1$, the WPOM response function is in good agreement with the POM one. To corroborate this statement we compute the analytic expression of the heat flux and the stress in the stationary limit $\xi \to 0$. This is possible because we can compute the explicit expressions of the

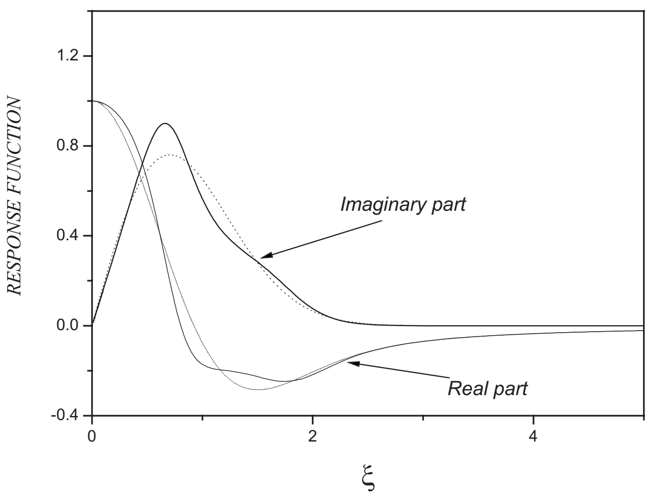

FIG. 1. Kinetic (solid line), WPOM fluid (dotted line), and POM (dashed line) response functions as a function of the dimensionless parameter $\xi=\frac{\omega}{\sqrt{2}kv_t}$.

collisionless continued fractions in the stationary limit. The relevant continued fractions F_1 and F_2 can be expressed as infinite products which can be computed with the Stirling formula,

$$F_1(\nu\to 0,\omega\to 0)=\frac{6}{\pi}\frac{y^{-1/2}}{\sqrt{2}|k|v_t}, \tag{81}$$

$$F_2(\nu\to 0,\omega\to 0)=\frac{5\pi}{8}\frac{y^{-1/2}}{\sqrt{2}|k|v_t}. \tag{82}$$

Substituting expressions (81) and (82) into the components of the DF and calculating the transport quantities from Eqs. (78) and (79) we obtain

$$q_x^{WPOM}(k)=-\frac{32}{(2\pi)^{3/2}}n_0T_0v_t\left(\frac{ik}{|k|}\frac{n}{n_0}\right)-\frac{48}{(2\pi)^{3/2}}n_0T_0v_t\left(\frac{ik}{|k|}\frac{T}{T_0}\right)-\frac{5}{2}n_0T_0v_t\left(\frac{V}{v_t}\right), \tag{83}$$

$$\Pi_{xx}^{WPOM}(k)=-n_0T_0\left(\frac{n}{n_0}\right)-n_0T_0\left(\frac{T}{T_0}\right), \tag{84}$$

and

$$q_x^{POM}(k)=-\frac{21}{8}\sqrt{\frac{2}{\pi}}n_0T_0v_t\left(\frac{ik}{|k|}\frac{n}{n_0}\right)-\frac{27}{8}\sqrt{\frac{2}{\pi}}n_0T_0v_t\left(\frac{ik}{|k|}\frac{T}{T_0}\right)-\frac{5}{2}n_0T_0v_t\left(\frac{V}{v_t}\right), \tag{85}$$

$$\Pi_{xx}^{POM}(k)=-n_0T_0\left(\frac{n}{n_0}\right)-n_0T_0\left(\frac{T}{T_0}\right). \tag{86}$$

We can observe the very good agreement between the WPOM and the POM in the quasistatic limit. The worst agreement is given by the thermal conductivities where the relative error is about 13%.

In addition, in Fig. 1 we can see in the intermediate range of ξ (or $\frac{\omega}{k}\sim v_t$), significant departures of the WPOM fluid response function from the POM one. We note that this regime is particularly important in neutral gases since many physical phenomena occur in this ξ range (sound waves, shock waves, etc.). Therefore we expect that in this regime the WPOM should give poor results even though we have kept all the anisotropic components of the DF.

IV. LINEAR STABILITY AND REGULARIZED SUPER-BURNETT EQUATIONS

It is well known that the NSE are stable in the whole wave-number range. But it is clear that this does not mean that the approximation of NS can be applied for arbitrary Knudsen number. Actually, its validity is typically limited to $K<10^{-3}$. In contrast to the NSE, the SBE suffer from instabilities at high wave numbers. This instability is frequently called in the literature as the Bobylev instability [27–30].

In this section we study the stability of the SBE. We limit our analysis as usual to the time stability; i.e., we consider a real wave number k and a complex frequency $\omega=\omega_r+i\gamma$. The system is stable means that any disturbance is damped and thus $\gamma<0$. If $\gamma>0$, the small disturbance grows exponentially and the numerical scheme to solve the set of EFD becomes instable. We study the problem by means of the EFD written if the Fourier space using dimensionless parameters,

$$-i\Omega\frac{n}{n_0}+iK\frac{V}{v_t}=0, \tag{87}$$

$$\left(iK+\frac{4}{3}iK^3\right)\frac{n}{n_0}+\left(-i\Omega+\frac{4}{3}K^2-\frac{16}{9}K^4\right)\frac{V}{v_t}+iK\frac{\delta T}{T_0}=0, \tag{88}$$

$$\frac{2}{3}K^4\frac{n}{n_0}+\left(\frac{2}{3}iK+\frac{2}{9}iK^3\right)\frac{V}{v_t}+\left(-i\Omega+\frac{5}{3}K^2-\frac{25}{9}K^4\right)\frac{\delta T}{T_0}=0, \tag{89}$$

where $\Omega=\frac{\omega}{\nu}$ and K is the Knudsen number. The resulting dispersion relation reads

$$i\Omega^3+\left(-3K^2+\frac{17}{9}K^4\right)\Omega^2-\left[\frac{20}{9}(-K^2+K^4)\left(-K^2+\frac{5}{2}K^4\right)+\frac{5}{3}K^2+\frac{14}{9}K^4\right]i\Omega-\frac{5}{3}\left(K^2+\frac{4}{3}K^4\right)\left(-K^2+\frac{5}{2}K^4\right)-\frac{2}{3}K^6=0. \tag{90}$$

The numerical computation of the dispersion relation (90) is a straightforward problem. For the purpose to test the stability of the EFD, we give in Figs. 2 and 3 only the imaginary part of the frequency ω (or the real part of the frequency $-i\Omega$) as a function of the wave number. The dispersion rela-

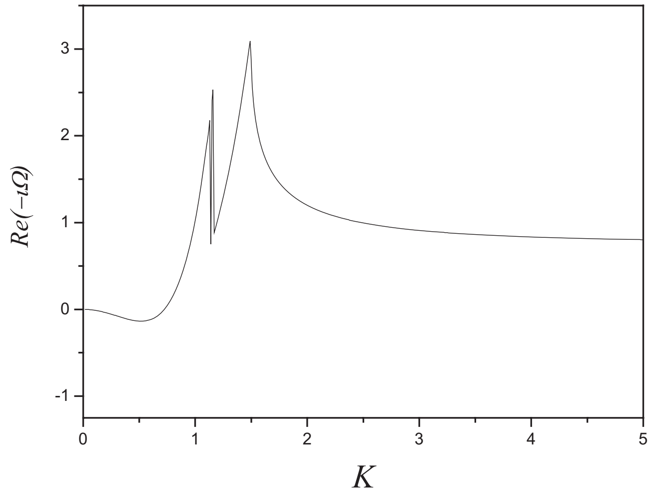

FIG. 2. Real part of the first root of the dispersion relation (90). The domain of stability is restricted to $K<0.72$.

tion (90) has three roots that we must consider for the stability tests. We find that the SB model yields unstable scheme for Knudsen numbers greater than $K>0.72$. These instabilities are not due to the lack of accuracy of the transport coefficients at a given order since the CE expansion is exact at each order but rather to the method used to perform the truncation of the CE expansion in the Knudsen number. This truncation should be performed on the basis of physical considerations. Indeed, it would be necessary for transport coefficients to verify the property of positivity.

To well clarify this point, we first study the stability of the Burnett equations using in the EFD the closure relations (44), (45), (51), and (52). The results of the test of stability are displayed in Fig. 4 where we see that the model is stable for arbitrary values of the Knudsen number. To explain the stability we note that in Eqs. (44) and (45) the thermal conductivity and the viscosity coefficient correspond to the NS transport coefficients and they are constant positive coefficients. The second terms (51) and (52) are diffusion terms with their usual meaning. This result shows that when the

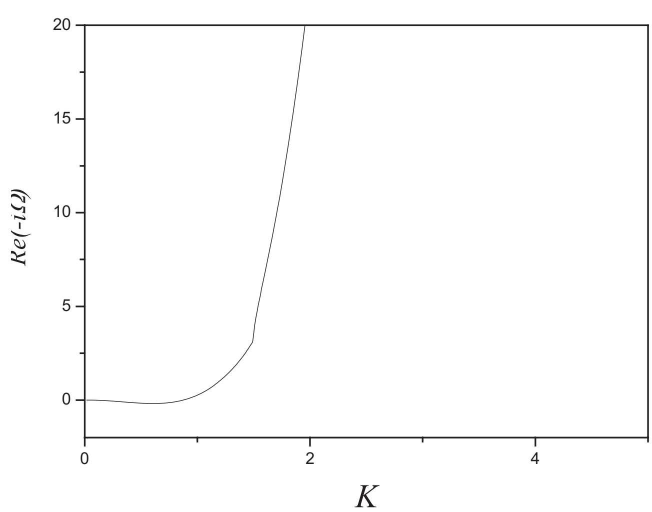

FIG. 3. Real part of the double root of the dispersion relation (90). The domain of stability is restricted to $K<0.87$.

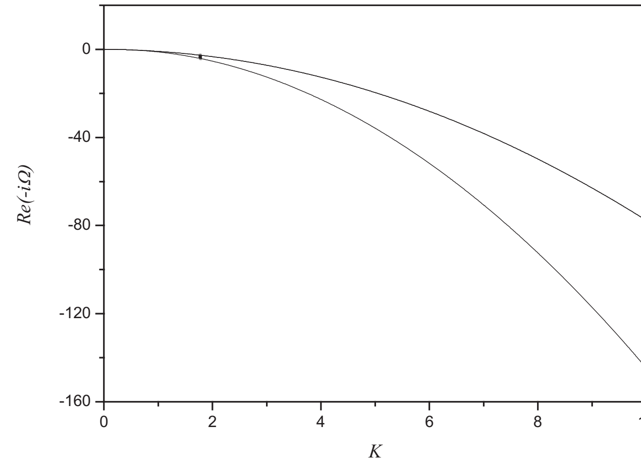

FIG. 4. Real part of the roots of the dispersion relation (90) restricted to the Burnett approximation.

transport coefficients verify the property of positivity (i.e., their physical meaning is well established) the system of EFD is stable.

Now we consider the SB model defined by the Burnett transport coefficients and the third-order terms (58) and (59), which we rewrite as

$$q_x^{SB}(K) = -in_0 v_t KT\left(\frac{5}{2} - \frac{25}{6}K^2\right) + \frac{1}{3}n_0 T_0 K^2 V - iT_0 v_t K^3 n, \tag{91}$$

$$\Pi_{xx}^{SB}(K) = -i\frac{n_0 T_0}{v_t}KV\left(\frac{4}{3} - \frac{16}{9}K^2\right) + \frac{4}{3}T_0 K^2 n. \tag{92}$$

The first two terms in Eqs. (91) and (92) represent the heat flux and the viscous stress defined by nonlocal transport coefficients (i.e., these coefficients depend on k). In the usual range of applicability of the SBE (typically $K<0.1$) these coefficients as they should are positive. However for the numerical solution of nonlinear EFD, they have to be positive for large values of K in order to avoid numerical instabilities. Following the method of regularization of the Burnett hydrodynamics by Padé approximants [31–35], we can rewrite these coefficients in a more suitable mathematical expression,

$$\Lambda = 5/2\left(1 + \frac{5}{3}K^2\right)^{-1}, \tag{93}$$

$$\eta = 4/3\left(1 + \frac{4}{3}K^2\right)^{-1}. \tag{94}$$

The third term in Eq. (91) cannot be used alone because it introduces a divergence for large Knudsen numbers. We must either drop it or calculate the next higher order term and build with these two terms a Padé approximant which fulfills the desirable mathematical properties. In this work for simplicity, we just consider the first approximation; i.e., we use the Padé approximants (93) and (94) and drop the third term in Eq. (91). The test of stability (see Fig. 5) shows that

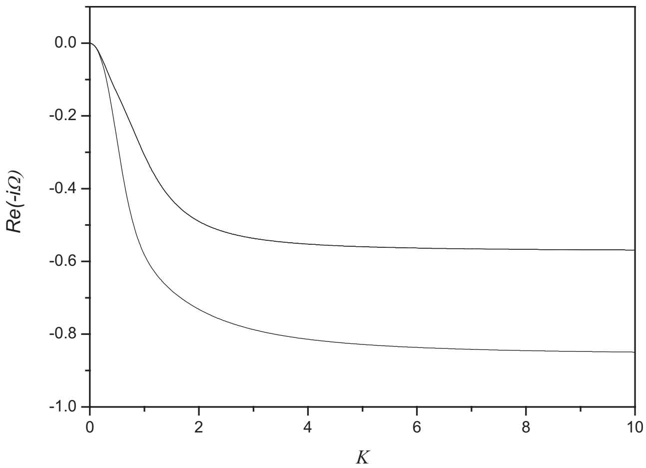

FIG. 5. Real part of the roots of the dispersion relation (90) for the POM. The second and the third terms in Eqs. (91) and (92) are dropped and Padé approximants (93) and (94) were used.

the EFD are stable for arbitrary Knudsen number. In addition, to do the link between the positivity of the thermal conductivity and the viscosity coefficient, and the stability of the EFD, we have kept in Eqs. (91) and (92) only the first terms and performed the test of stability. We obtained that the system is unstable for $K>0.72$. We can see that this value agrees well with the limit of positivity of these two coefficients which is $K<\sqrt{3/5}$.

On the other hand we note that the Padé approximants used above led to a simple expression of the transport relations in the real space. Transforming in the real space, the SB transport expressions with the use of Padé approximants (93) and (94), it results

$$q_x^{SB}(x) = \int \left(-\frac{5}{2}\frac{n_0 v_t^2}{\nu}\frac{\partial T}{\partial x'}\right)\frac{\exp\left(-\frac{|x-x'|}{\lambda_\Lambda}\right)}{2\lambda_\Lambda}dx' - \frac{1}{3}\frac{n_0 T_0 v_t^2}{\nu^2}\frac{\partial^2 V}{\partial x^2}, \tag{95}$$

$$\Pi_{xx}^{SB}(x) = \int \left(-\frac{4}{3}\frac{n_0 T_0}{\nu}\frac{\partial V}{\partial x'}\right)\frac{\exp\left(-\frac{|x-x'|}{\lambda_\eta}\right)}{2\lambda_\eta}dx' - \frac{4}{3}\frac{T_0 v_t^2}{\nu^2}\frac{\partial^2 n}{\partial x^2}, \tag{96}$$

where $\lambda_\Lambda \approx 0.63$ and $\lambda_\eta \approx 1.15$ are delocalization lengths and the factors before the exponentials correspond to the local NS expressions. We can see that the first terms in Eqs. (95) and (96) have the desirable properties; i.e., when the delocalization lengths tend to zero, the kernel behaves as a Dirac function and we retrieve the Navier-Stokes local expressions. In addition, owing to the nonlocal effects, the delocalization kernels tend to reduce the heat flux and the stress and to spread them spatially.

V. SUMMARY

The purpose of this work is to present an approach to solve the kinetic equation for neutral gases based on the projection operator technique to ensure the conservation laws and, on the continued fractions to incorporate the contributions from all the Legendre modes. This method was successfully used in Ref. [13] to calculate in the whole collisionality range the dispersion relation of sound waves. The results obtained by this method in the weakly collisional range up to the third order in the Knudsen number match exactly with the CE expansion (SB equations). The role of the projection operators is also investigated by calculating the transport without the contribution of these operators. It has been shown that their contribution is not negligible from the third order in the Knudsen number. Furthermore, in the collisionless limit, significant departure from the exact results is pointed out. The collisionless range is also used as a benchmark to test the exactness of the POM. We have found that both the kinetic and the POM fluid response functions match perfectly in the whole ξ range.

The stability analysis of the SBE is also investigated. It has been shown that the EFD are unstable beyond the Burnett order. This confirms that this instability is inherent to the asymptotic expansion of the distribution function in the Knudsen number. The correlation between the positivity of the transport coefficients and the domain of stability of the SB equations is emphasized. Finally nonlocal transport coefficients with the use of the Padé approximants which guarantee the stability of the linear EFD are also proposed.

In this work, we have presented the solution of the Boltzmann equation modeled with the BGK operator, which are valid in the whole collisionality range. The same approach can be used on other forms of perturbed kinetic equations. To inverse the linearized Boltzmann operator we expect some technical problems that we shall work out.

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